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Computing the Singular Behavior of Solutions of Cauchy Singular Integral Equations with Variable Coefficients

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Abstract—A numerical technique to determine the singular behavior of the solution of Cauchy singular integral equations (CSIE) with variable coefficients is proposed. The fundamental solution, which is a solution of the corresponding homogeneous equation, is constructed by a quadrature-collocation scheme. This leads to a system of nonlinear equations to approximate the exponents of singularity. Newton's method has been found to yield a useful approximation to these exponents. Once we have numerically obtained the weight function which determines the behavior of the solution of Cauchy singular integral equation, it can be used to solve a variety of nonhomogeneous CSIE with variable coefficients.

Keywords—Singular integral equation, Fundamental solution, Nonlinear equations.

1. INTRODUCTION

Singular integral equations with a Cauchy principal value arise frequently in mixed boundary value problems for partial differential equations. In many of these problems, auxiliary functions may have singularities of a prescribed nature. Often these equations are not amenable to a closed form solution. For such equations, a prior knowledge of singular behavior is helpful in devising appropriate numerical integration schemes. The classical theory for such integral equations is based on the properties of sectionally holomorphic functions (functions which are analytic in a complex plane cut along a line segment). However, there is no analogous theory to deal with multidimensional integral equations. This paper attempts to augment the classical theory by suggesting a numerical approach to the problem of finding the singular behavior near the endpoints of a slit by developing the technique of numerical integration proposed by Srivastav [1] for Cauchy singular integral equations with constant 'coefficients' to those with variable coefficients. Only the one-dimensional integral equation is discussed here. Without loss of generality we may take the domain for integration and the equation as $(-1, 1)$.

Our approach is pretty simple. Since the dominant singularity is at the endpoints, we express the fundamental function (also called the weight function) as

$$w(x) = (1-x)^{p(x)}(1+x)^{q(x)}.$$

We then replace the integral by Gauss-Chebyshev and Lobatto-Chebyshev quadrature formulae. Collocation at an appropriate set of nodes then leads to a nonlinear system of algebraic equations

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX for the values of $p(x)$ and $q(x)$. We solve this system of equations using a Newton-Raphson method. This weight function is then used to solve the integral equation using well-known techniques.

Since Gaussian quadrature in the limit is the Riemann sum, one would expect the technique to converge even when the Hölder continuity condition is violated.

The paper is organized in 6 sections as follows: the fundamental solution for variable coefficient CSIE is constructed as an exponentiation form in Section 2. A nonlinear equations system is built to determine the singular behavior of the solution numerically by the Gauss-Lobatto-Chebyshev method in Section 3. Newton's method has been found to yield a useful approximation to the exponential representation of the weight function, as described in Section 4. The numerical examples are presented in Section 5. Section 6 contains concluding remarks.

2. THE FUNDAMENTAL SOLUTION FOR VARIABLE COEFFICIENT CSIE

A Cauchy-type singular integral equation has the form

$$a(x)\phi(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{\phi(t)\,dt}{t-x} + \int_{-1}^{1}k(x,t)\phi(t)\,dt = f(x), \qquad 0 < x < 1. \tag{1}$$

Here, $a(x)$, $b(x)$, $k(x,t)$, $f(x)$ are known real functions, and $\phi(x)$ is the unknown function. Following [2], we assume that $a(x)$ and $b(x)$ are Hölder continuous on $[-1,1]$, and likewise $k(x,t)$ is Hölder continuous in each variable on $[-1,1]$. We require also that $a^2(x)+b^2(x)$ never vanish on $(-1,1)$.

Solutions to equation (1) are sought in the space H^* of functions Hölder continuous in every closed subinterval of $(-1,1)$. Functions with a singularity at the endpoints ± 1 are admissible as solutions. We shall assume that the singularity is a weak singularity, i.e., the functions are integrable in the usual sense. The integral term containing $k(x,t)$ does not effect the nature of singularity at the endpoints. The qualitative behavior is the same as that of the homogeneous equation

$$a(x)\phi(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{\phi(t)\,dt}{t-x} = 0, \qquad -1 < x < 1. \tag{2}$$

Let

$$G(x) = \frac{a(x)-ib(x)}{a(x)+ib(x)}.$$

The fundamental solution[1] $Z(x)$ of equation (2) is now expressible as

$$Z(x) = (1-x)^{\lambda_1}(1+x)^{\lambda_2}\exp\left(\frac{1}{2\pi}\int_{-1}^{1}\frac{\arg G(t)}{t-x}\,dt\right). \tag{3}$$

Here,

$$\begin{aligned}
\ln G(t) &= \ln|G(t)| + i\arg G(t), \qquad -\pi < \arg G(t) < \pi,\\
\alpha_1 &= \frac{-1}{2\pi i}\ln G(-1),\\
\alpha_2 &= \frac{1}{2\pi i}\ln G(1),
\end{aligned}$$

$-1 < \alpha_j + \lambda_j < 1$, λ_j are integers, $\chi = -(\lambda_1+\lambda_2)$, χ is called the index of the equation (1). When $a(x)$ and $b(x)$ are both constants, $Z(x)$ may be expressed as

$$(1-x)^{-\sigma}(1+x)^{\sigma-1}.$$

[1]Numerical integration is in general not useful at this stage.

As a first step to determine the singular behavior of the solution of the CSIE's with variable coefficients, we find a similar expression for the fundamental solution.

THEOREM 1. *The fundamental solution of the homogeneous equation can be written in the form of* $Z(x) = (1-x)^{p(x)}(1+x)^{-\chi-p(x)}$.

PROOF. Let

$$\left(\frac{1-x}{1+x}\right)^{c(x)} = \exp\left(\frac{1}{2\pi}\int_{-1}^{1}\frac{\arg G(t)}{t-x}\,dt\right). \tag{4}$$

Obviously,

$$c(x) = \left(\frac{1}{2\pi}\int_{-1}^{1}\frac{\arg G(t)}{t-x}\,dt\right)\left(\ln\left(\frac{1-x}{1+x}\right)\right)^{-1}.$$

We obtain

$$Z(x) = (1-x)^{\lambda_1}(1+x)^{\lambda_2}\left(\frac{1-x}{1+x}\right)^{c(x)}, \tag{5}$$

so that we have

$$p(x) = \lambda_1 + c(x), \qquad -\chi - p(x) = \lambda_2 - c(x).$$

The proof is complete. ∎

The singular behavior of the solution is determined by function $p(x)$.

3. NUMERICAL DETERMINATION OF FUNDAMENTAL FUNCTION

We now derive algebraic equations to find an approximation to $p(x)$ numerically. While we use a different quadrature formula, in principle, it is the same as that of Srivastav [1]. We seek functions $p(x)$ which satisfy the equation

$$a(x)(1-x)^{p(x)}(1+x)^{-\chi-p(x)} + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{(1-t)^{p(t)}(1+t)^{-\chi-p(t)}}{t-x}\,dt = 0. \tag{6}$$

If we use the Gauss-Chebyshev quadrature formula and collocation at the zeros of $U_{n-1}(s)$ for equation (6), we get for $j = 1, 2, \dots, n-1$ the system of $(n-1)$ nonlinear equations:

$$a(s_j)(1+s_j)^{-\chi}\left(\frac{1-s_j}{1+s_j}\right)^{p(s_j)} + \frac{b(s_j)}{n}\sum_{k=1}^{n}\frac{(1+t_k)^{1-\chi}}{t_k - s_j}\left(\frac{1-t_k}{1+t_k}\right)^{p(t_k)+1/2} = 0. \tag{7}$$

Here $\{t_k\}$, $\{s_j\}$ are zeros of $T_n(x)$ and $U_{n-1}(x)$, respectively.

Similarly, using the Lobatto-Chebyshev quadrature formula and collocation at the zeros of $T_n(x)$, we get, for $n = 1, 2, \dots, n$,

$$a(t_k)(1+t_k)^{-\chi}\left(\frac{1-t_k}{1+t_k}\right)^{p(t_k)} + \frac{b(t_k)}{n}\sum_{j=1}^{n-1}\frac{(1+s_j)^{1-\chi}}{s_j - t_k}\left(\frac{1-s_j}{1+s_j}\right)^{p(s_j)+1/2} = 0. \tag{8}$$

Combining the equations systems (7) and (8), we get $(2n-1)$ nonlinear equations about $(2n-1)$ unknown $\{p(s_j),\, p(t_k)\}$, which may be written in the vector form as

$$F(P) = 0, \tag{9}$$

where

$$P = (p(s_1), p(s_2), \dots, p(s_{n-1}), p(t_1), p(t_2), \dots, p(t_n))^{\top}.$$

4. NEWTON'S METHOD FOR THE NONLINEAR EQUATIONS

To solve this nonlinear equation system (9), we use the standard Newton's method

$$\frac{DF}{DP}\left(P^{(i)}\right)\Delta P^{(i)} = -F\left(P^{(i)}\right), \qquad P^{(i+1)} = P^{(i)} + \Delta P^{(i)}, \tag{10}$$

where $i = 0, 1, \dots,$ are the Newton iterative steps, and

$$\frac{DF}{DP}\left(P^{(i)}\right) = \begin{pmatrix} c_1 & & & a_{1\,1} & \cdots & a_{1\,n} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & c_{n-1} & a_{n-1\,1} & \cdots & a_{n-1\,n} \\ b_{1\,1} & \cdots & b_{1\,n-1} & d_1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ b_{n\,1} & \cdots & b_{n\,n-1} & & & d_n \end{pmatrix}, \tag{11}$$

$$c_j = a(s_j)(1+s_j)^{-\chi}\left(\frac{1-s_j}{1+s_j}\right)^{p(s_j)} \ln\left(\frac{1-s_j}{1+s_j}\right), \tag{12}$$

$$d_k = a(t_k)(1+t_k)^{-\chi}\left(\frac{1-t_k}{1+t_k}\right)^{p(t_k)} \ln\left(\frac{1-t_k}{1+t_k}\right), \tag{13}$$

$$a_{jk} = \frac{b(s_j)}{n}\frac{(1+t_k)^{1-\chi}}{t_k - s_j}\left(\frac{1-t_k}{1+t_k}\right)^{p(t_k)+.5} \ln\left(\frac{1-t_k}{1+t_k}\right), \tag{14}$$

$$b_{kj} = \frac{b(t_k)}{n}\frac{(1+s_j)^{1-\chi}}{s_j - t_k}\left(\frac{1-s_j}{1+s_j}\right)^{p(s_j)+.5} \ln\left(\frac{1-s_j}{1+s_j}\right). \tag{15}$$

In her Ph.D. Thesis, Li [3] has shown that Newton's method converges[2] quadratically for each n. Our computational experience indicates that as n is increased, the discrete solution gets closer to the analytical solution when known.

5. NUMERICAL EXAMPLE

Since there are no benchmark problems, we have solved a variety of problems. We give here the numerical results for an equation with known solution. For

$$a(x) = \sin\left(\frac{\pi x}{2}\right), \quad b(x) = \cos\left(\frac{\pi x}{2}\right), \qquad \chi = 1,$$

we have, with $\phi(0) = 1$,

$$\phi(x) = \cos\left(\frac{\pi x}{2}\right)(1-x)^{(x-1)/2}(1+x)^{-(1+x)/2},$$

$$p(x) = \frac{x-1}{2} + \left[\ln\left(\frac{1-x}{1+x}\right)\right]^{-1} \ln\left[\cos\left(\frac{\pi x}{2}\right)\right].$$

The quadrature and collocation nodes are zeros of U_{61} corresponding to $n = 31$. To display the results, we have renamed them x_m, $m = 1, 2, 3, \dots, 61$ from left to right. Table 1 shows the exact and computed values of $p(x)$ as well as their difference. Table 2 does the same thing for the solution of the homogeneous equation. We note that the error is less than 10^{-6} after only seven iterative steps . We do not have analytical error estimates and do not know how the error will propagate in the solution of nonhomogeneous equations, but the computational evidence seems to be positive.

[2]Convergence of Newton's method for a fixed system does not imply the convergence of the numerical computation of the solution.

Table 1. Exact and computed values of $p_n(x)$ of Example 1.

m	x_m	Exact Sol. $p_n(x_m)$	Num. Sol. $p_n(x_m)$	Error
1	−0.998716507	−1.843714424	−1.843725177	−0.000010753
5	−0.968077117	−1.710297555	−1.710298868	−0.000001313
9	−0.897804533	−1.576505422	−1.576505894	−0.000000472
13	−0.790775723	−1.422092145	−1.422092370	−0.000000225
17	−0.651372460	−1.245306445	−1.245306549	−0.000000104
21	−0.485301930	−1.048381395	−1.048381391	0.000000004
25	−0.299363081	−0.835703258	−0.835703022	0.000000236
29	−0.101168271	−0.613039862	−0.613037209	0.000002652
34	0.151427837	−0.330708700	−0.330708175	0.000000525
38	0.347305316	−0.109947761	−0.109947830	−0.000000070
42	0.528964073	0.099289389	0.099289272	−0.000000116
46	0.688966978	0.291522629	0.291522472	−0.000000157
50	0.820763492	0.462816664	0.462816411	−0.000000253
54	0.918957849	0.611670776	0.611670260	−0.000000515
58	0.979529962	0.741655611	0.741653997	−0.000001614
61	0.998716513	0.843714518	0.843725316	0.000010798

Table 2. Exact and computed values of solution $\phi_n(x)$.

m	x_m	Exact Sol. $\phi_n(x_m)$	Num. Sol. $\phi_n(x_m)$	Error
32	0.050649225	0.995558461	0.995557583	−0.000000878
33	0.101168380	0.982350834	0.982350297	−0.000000537
35	0.201298581	0.931232296	0.931232027	−0.000000269
37	0.299363186	0.851804588	0.851804400	−0.000000188
39	0.394355919	0.751665000	0.751664845	−0.000000155
41	0.485302026	0.639612647	0.639612505	−0.000000142
43	0.571268278	0.524363336	0.524363199	−0.000000137
45	0.651372543	0.413479919	0.413479783	−0.000000137
47	0.724792845	0.312688094	0.312687956	−0.000000138
49	0.790775790	0.225625213	0.225625072	−0.000000140
51	0.848644305	0.153963218	0.153963075	−0.000000143
53	0.897804581	0.097783767	0.097783621	−0.000000147
55	0.937752166	0.056069550	0.056069400	−0.000000150
57	0.968077144	0.027199911	0.027199761	−0.000000150
59	0.988468340	0.009383509	0.009383368	−0.000000141
61	0.998716513	0.001013465	0.001013384	−0.000000080

6. CONCLUDING REMARKS

Based on a specific singular representation of the weight function, we propose a numerical method for solving CSIE. Gaussian quadrature and collocation is used to determine the exponents of singularity at the endpoints. Newton's method has been found to yield a useful approximation to the exponential representation of the weight function. Even when Hölder continuity condition is violated, the numerical scheme appears to be effective for problems where the solution exists. Once we have numerically obtained the weight function which determines the behavior of the solution of Cauchy singular integral equation, one can successfully use it for solving a variety of nonhomogeneous CSIE with variable coefficients. Moreover, since the approach suggested here does not use complex analysis explicitly, it may be possible to extend it to multidimensional singular integral equations. Our paper may be viewed as the first step towards a real variable theory for Cauchy singular integral equations at an elementary level.

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